SOME METHODS OF NUMERICAL SOLUTION OF A PROBLEM IN THREE-DIMENSIONAL UNSTEADY PERCOLATION

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A study has been made of the boundary-value problem in the region $D(0 \leqslant x_1 \leqslant a,\ 0 \leqslant x_2 \leqslant a)$:

$$\frac{\partial h}{\partial t} = \frac{k}{m}\left\{\frac{\partial}{\partial x_1}\left(h\frac{\partial h}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(h\frac{\partial h}{\partial x_2}\right)\right\} \qquad (0 \leqslant t \leqslant T), \tag{1}$$

$$h(x_1, x_2, 0) = H_0, \quad h(0, x_2, t) = h(x_1, 0, t) = H_1, \quad h(x_1, a, t) = h(a, x_2, t) = H_0\ . \tag{2}$$

Equations (1), (2) describe the unsteady motion of ground water from a free surface in a layer of finite depth over a horizontal impermeable base without infiltration or evaporation from the free surface being taken into account; k and m in (1) are used to denote the coefficients of percolation and water loss of the soil, $h(x_1, x_2, t)$ is the ground water head at point x_1, x_2 at time t. Two implicit difference schemes were studied for this problem, which has a slight nonlinearity.

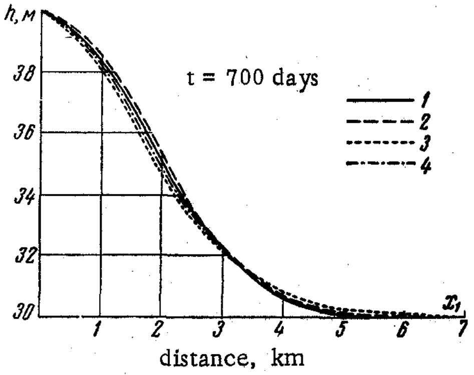

Fig. 1

The first scheme is known as the local one-dimensional method of variable directions [1]; its theory is quite well developed. For conditions (1), (2) this scheme is as follows. At each moment of time $t_{j+i/2}$, $i = 1, 2$, $j = 0, 1, \ldots, K$, $\tau = T/K$, we solve the equation

$$\frac{\partial h_1}{\partial t_1} = \frac{\partial}{\partial x_i^*}\left(h_1\frac{\partial h_1}{\partial x_i^*}\right),$$
$$h_1 = \frac{h}{H_0}, \quad x_i^* = \frac{x_i}{A}, \quad t_1 = \frac{t}{B}, \quad B = \frac{mA^2}{k\langle h\rangle}, \quad A = 10^3 \tag{3}$$

(< h > is some mean value of the head).

As boundary conditions we use the values of the boundary functions at the points of intersection of straight lines parallel to the $0x_i$-axis and the boundary of the region of integration; as initial values we use the values obtained in the computations for the preceding layer. The second variable x_j ($j \neq i$) enters the equation as a parameter. The solution of the finite-difference analog of Eq. (3) is denoted $z = z(x_1\ x_2\ t)$; in this case

$$z(x_1^{i_1}, x_2^{i_2}, t_j) = z_{i_1i_2}^{j} = z_{i_1i_2j}\ .$$

The local one-dimensional, three-point, second-order difference scheme [1] for Eq. (3) has the form

$$\frac{1}{\tau}\left(z_{i_1i_2}^{j+1/2} - z_{i_1i_2}^{j}\right) = \Lambda_1 z_{i_1i_2}^{j+1/2}\ ,$$
$$\frac{1}{\tau}\left(z_{i_1i_2}^{j+1} - z_{i_1i_2}^{j+1/2}\right) = \Lambda_2 z_{i_1i_2}^{j+1}\ . \tag{4}$$

Here Λ_i is an operator approximating the initial differential operator

$$\frac{\partial}{\partial x_i}\left(h\frac{\partial h}{\partial x_i}\right) \to \Lambda_i z = \frac{z^{+1_i}(z^{+1_i} - z) - z(z - z^{-1_i})}{h_i^2}\ , \qquad \begin{aligned} x^{\pm mi} &= x_i \pm m h_i \\ z^{\pm mi} &= z(x^{\pm mi}) \end{aligned}$$

and h_i is the space variable interval.

The second scheme was developed on the basis of the method proposed in [2] for the heat conduction equation. Equation (1) is reduced to the form

$$\frac{\partial h^2}{\partial t} = \frac{kh}{m}\left[\frac{\partial^2 h^2}{\partial x_1^2} + \frac{\partial^2 h^2}{\partial x_2^2}\right]. \tag{5}$$

In scheme 2, in addition to the values of the approximating function z, in the j-th and (j + 1)-th layers with respect to time, in accordance with [2], we introduce some intermediate solution $Z_{i_1i_2,\ j+1}$ of the problem (which corresponds to the introduction of an intermediate layer j + ½ into scheme 1). Equation (5) is replaced by the following system of finite-difference relations:

$$\frac{1}{\tau}(Z_{j+1}^{2*} - z_j^2) = \frac{k}{m} z_j (\Delta_{x_1}{}^2 Z_{j+1}^2 + \Delta_{x_2}{}^2 z_j),$$
$$\frac{1}{\tau}(z_{j+1}^2 - Z_{j+1}^2) = \frac{k}{m} Z_j (\Delta_{x_2}{}^2 z_{j+1}^2 - \Delta_{x_2}{}^2 z_j^2), \qquad \Delta_{x_i}{}^2 z_j = \frac{z_{i_{i+1},j} - 2z_{i_i,j} + z_{i_{j-1},j}}{h_i^2} \tag{6}$$

The convergence and the stability of the corresponding linear system was demonstrated in [2]. The stability of the nonlinear system (6) was checked experimentally.

The behavior of schemes 1 and 2 was studied for a linearized Eq. (1) of the form

$$\frac{\partial h_1}{\partial t_1} = \frac{\partial^2 h_1}{\partial X_1^2} + \frac{\partial^2 h_1}{\partial X_2^2}, \tag{7}$$

for which an exact solution is known [3], with the initial and boundary conditions stipulated in [4], where the problem (7), (2), is solved as an example, using an explicit difference scheme, for k = 5 m/day, m = 0.06, < h > = H_0 = 30 m, H_1 = 40 m, τ = 100 days, $h_1 = h_2$ = 1000 m, a = 10 000 m. The exact solution of (7), (2) is expressed by the formula

$$h(x_1, x_2, t) = H_1 - (H_1 - H_0)\,\Phi(x_1/2a\sqrt{t})\,\Phi(x_2/2a\sqrt{t}).$$

Here Φ is the probability integral. This formula and (4) and (6) were used in making computations with the above-mentioned initial and boundary conditions. Figure 1 shows diagonal sections of the tables of ground water levels at the time t = 700 days. For comparison we show computations using the explicit scheme given in [4]. In Fig. 1, curve 1 corresponds to the exact solution and curve 2 to the explicit scheme; curve 3 corresponds to the second scheme and curve 4 to the first scheme. The closest coincidence with the exact solution is given by the computations for the first scheme. The explicit method might be preferred due to the simplicity of the computations; however, as is well known, it has a serious shortcoming — the need for severe constraints on the grid. In [4], for example, $\tau/h_i^2 = 10^{-4}$. Moreover, in problems of this type it is necessary to compute the behavior of the free surface of the ground water flow over long periods of time, and therefore the limitation on the time interval is particularly restrictive. Implicit methods are free of this shortcoming and are convenient from the programming point of view. Both schemes are applicable if $h(x_1, x_2, t) \geqslant c_0 > 0$.

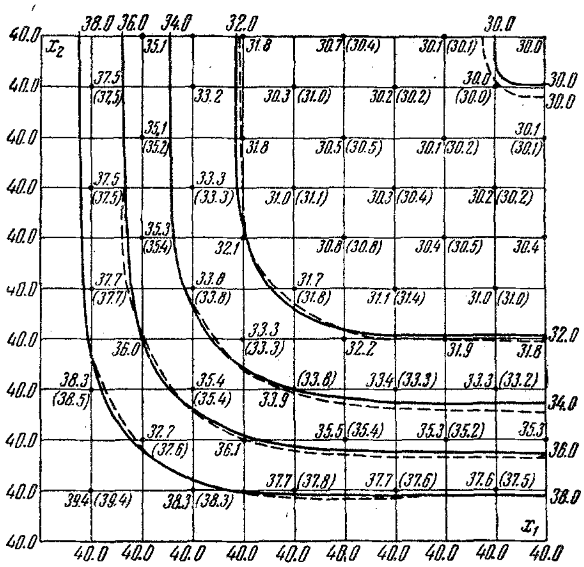

Fig. 2

The nonlinear equations (4), (6) with initial and boundary conditions (2) were solved by iterations of the following form. As the initial approximation in the coefficients we selected values of the function from the preceding layer and used the pivot method to compute the first approximation of the required function. This value was likewise entered into the coefficients and the second iteration was computed, and so forth, until the stipulated accuracy was attained.

Figure 2 shows lines representing the surface levels of the ground water, computed using schemes 1 and 2 (continuous and broken lines, respectively); the levels at the individual nodes of the grid are also noted (the values in parentheses corresponds to scheme 2).

The discrepancies in the values given by the two schemes are small and decrease with time. This suggests the use of scheme 2, which in the solution of (7), (2) involved fewer computations.

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